LECTURE 21 LINEARIZATION

When we find the derivative of a function, as a function, we are basically saying that at each point x, the function looks like its tangent line. This is a fair approximation as we move along. But moving along incurs too much computational resource, namely, you are required to evaluate the derivative at every point. What if our domain of interest is very small, such that points far away from this domain is of no interest? Can we simply claim that, the function on this domain, can be simply approximated by a line, namely, the tangent line at the left most point of the domain? We miss by some error, certainly, but there is a plus side.

Suppose our function is

$$f(x) = \sin(x)\, e^{\cos^2(x)}$$

a very ugly looking function. The computational resource we need to compute the value of $f(x)$ at a specific x is very high – namely, $\sin(x)$ is costly, square root is costly, and exponential also is costly.

However, suppose we only want the function values on the interval $\left[0, \frac{\pi}{2}\right]$. Can we exchange evaluation error for computational simplicity, on this interval? Let's compute the tangent line, represented by $L(x)$, at $x = 0$, and claim that we can approximate all function values of $f(x)$ on this interval, by $L(x)$, with controllable error.

Note that

$$L(x) = mx + b$$

where $m = f'(0)$. How do find b? This is the equation of the tangent line at $(0, f(0))$. Therefore, we do point-slope form,

$$L(x) - f(0) = m(x - 0) \implies L(x) = f'(0)\, x + f(0).$$

In general, for the tangent line at $x = a$, and thus at the coordinate $(a, f(a))$, we find

$$L(x) - f(a) = f'(a)(x - a) \implies L(a) = f(a) + f'(a)(x - a).$$

We saw that $L(x)$ is the linearization of $f(x)$ at $x = a$. One **must** specify the point at which the linearization is about.

Example. Consider $f(x) = x^2$. Find the linearization $L(x)$ about $x = 1$. Are we underestimating/overestimating $f(x)$ using $L(x)$ on the interval $[1, 2]$?

Let's find the tangent line at $x = 1$.

$$f'(x) = 2x \implies f'(1) = 2.$$

Also, the function value is $f(1) = 1$. Thus,

$$L(x) = f(1) + f'(1)(x - 1) = 1 + 2(x - 1) = 2x - 1.$$

Let's plot the original function $f(x) = x^2$ and the linearization at $x = 1$, $L(x) = 2x - 1$. We see that $L(x)$ lies underneath $f(x)$ on $[1, 2]$, thus implying that it is always underestimating the original function.

Example. Let's now consider the function at the start of the class. Let's find its linearization at $x = 0$, where

$$f(x) = \sqrt{\sin(x)}e^{\cos^2(x)}$$

and decide how good the approximation is on the interval $\left[0, \frac{\pi}{2}\right]$.